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UPPER BOUND THEOREMS FOR HOMOLOGY MANIFOLDS

BY

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ABSTRACT

In this paper we prove the Upper Bound Conjecture (UBC) for some classes of (simplicial) homology manifolds: we show that the UBC holds for all odd-dimensional homology manifolds and for all $2k$-dimensional homology manifolds Δ such that

$$\beta_k(\Delta) \leq \sum\{\beta_i(\Delta)\colon i \neq k-2, k, k+2 \text{ and } 1 \leq i \leq 2k-1\},$$

where $\beta_i(\Delta)$ are reduced Betti numbers of Δ. (This condition is satisfied by $2k$-dimensional homology manifolds with Euler characteristic $\chi \leq 2$ when k is even or $\chi \geq 2$ when k is odd, and for those having vanishing middle homology.)

We prove an analog of the UBC for all other even-dimensional homology manifolds.

Kuhnel conjectured that for every $2k$-dimensional combinatorial manifold with n vertices, $(-1)^k(\chi(\Delta) - 2) \leq \binom{n-k-2}{k+1}/\binom{2k+1}{k}$. We prove this conjecture for all $2k$-dimensional homology manifolds with n vertices, where $n \geq 4k+3$ or $n \leq 3k+3$. We also obtain upper bounds on the (weighted) sum of the Betti numbers of odd-dimensional homology manifolds.

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1. Introduction

In this paper we prove several extensions of the upper bound theorem for convex polytopes. We start by briefly describing the history of the problem and give some definitions.

Let $C_d(n)$ be the cyclic d-polytope with n vertices. (That is, $C_d(n) = C_d(x_1, x_2, \dots, x_n) = \text{conv}\{v_1, v_2, \dots, v_n\} \subset \mathbf{R}^d$, where $x_1, x_2, \dots, x_n \in \mathbf{R}$ are all different and $v_i = (x_i, x_i^2, x_i^3, \dots, x_i^d)$.) It is well known that the combinatorial type of a cyclic d-polytope does not depend on the choice of $x_1, x_2, \dots, x_n$ and that all cyclic polytopes are simplicial ([22]).

Definition 1.1: The f-vector of a $(d-1)$-dimensional simplicial complex Δ is a vector $f(\Delta) = (f_{-1}, f_0, f_1, \dots, f_{d-1})$, where f_k denotes the number of k-dimensional faces of Δ.

The Upper Bound Conjecture (briefly, UBC) proposed by Motzkin in 1957 [12] asserts that if P is a (simplicial) d-polytope with $f_0 = n$ vertices, then for every $k = 1, \dots, d-1$

$$f_k(P) \leq f_k(C_d(n)).$$

This is obvious for $k \leq \lfloor d/2 \rfloor$, since every $k \leq \lfloor d/2 \rfloor$ vertices of the cyclic d-polytope form a face.

Recall that the Euler characteristic of a simplicial $(d-1)$-dimensional complex Δ, $\chi(\Delta)$, is defined by

$$\chi(\Delta) - 1 = \sum_{i=-1}^{d-1} (-1)^i f_i(\Delta) = \sum_{i=-1}^{d-1} (-1)^i \beta_i(\Delta),$$

where $\beta_i(\Delta)$ are reduced Betti numbers of Δ (that is, $\beta_i = \dim_{\mathbf{k}} \tilde{H}_i(\Delta; \mathbf{k})$ for some field $\mathbf{k}$; note that if Δ is a connected non-empty complex, then $\beta_{-1} = \beta_0 = 0$). Recall also that if Δ is a simplicial complex and F is a face of Δ, then the link of F, $\text{lk}\, F$, is a set $\{G \in \Delta : F \cap G = \emptyset, F \cup G \in \Delta\}$.

Definition 1.2: A simplicial complex Δ is a Eulerian complex, if for every face F of Δ (including the empty face) $\chi(\text{lk}\, F) = 1 + (-1)^{\dim \text{lk} F}$

In 1964, Klee conjectured that the assertion of the UBC holds for all Eulerian complexes and proved it for Eulerian complexes with sufficiently large number of vertices [8]. In 1970, McMullen [11] gave a complete proof of the UBC for polytopes, and in 1975 Stanley proved the UBC for arbitrary triangulations of spheres [17], [19].

In this paper we prove the UBC for several classes of homology manifolds: all odd-dimensional homology manifolds, all even-dimensional Eulerian homology manifolds (that is, homology manifolds that are Eulerian as simplicial complexes), and, in fact, an even larger class of homology manifolds. We also obtain the analog of the UBC for arbitrary even-dimensional homology manifolds.

Definition 1.3: A (finite) simplicial complex Δ (or, more precisely, the geometric realization X of Δ) is a homology manifold if for any $p \in X$ and any $i < \dim X$, $H_i(X, X-p) = 0$ and $H_{\dim X}(X, X-p) \cong \mathbf{Z}$, where $H_i(X, X-p)$ is the i-th relative singular homology with coefficients $\mathbf{Z}$.

Remark: (1) The link of any non-empty face of a homology manifold has the homology of the sphere (see [13]). Therefore, a homology manifold Δ is Eulerian iff $\chi(\Delta) = 1 + (-1)^{\dim(\Delta)}$.

(2) Note that any (triangulation of a) topological manifold is a homology manifold, but the converse is not true. In other words, there are the following relations between various classes of "manifolds":

homology manifolds $\supset$ triangulations of topological manifolds

($\supset$ combinatorial manifolds)

and all inclusions are strict.

THEOREM 1.4 (UBT for odd-dimensional homology manifolds): *Let Δ be a $(2k-1)$-dimensional homology manifold on n vertices. Then*

$$f_i(\Delta) \leq f_i(C_{2k}(n)) \quad \text{for } i = 1, \dots, 2k-1.$$

THEOREM 1.5: *The UBC holds for all $2k$-dimensional homology manifolds, such that*

$$\beta_k(\Delta) \leq \sum \{\beta_i(\Delta) : i \neq k-2, k, k+2 \text{ and } 1 \leq i \leq 2k-1\}.$$

Remark: Throughout this paper, $\beta_i(\Delta)$ are reduced Betti numbers of Δ, calculated with respect to any field of characteristic two or with respect to any other field $\mathbf{k}$ such that Δ is orientable over $\mathbf{k}$. Since a homology manifold is a Buchsbaum complex over any field and since the Euler characteristic does not depend on the choice of the field, this restriction does not affect our results. On the other hand, under such an assumption, $\beta_i = \beta_{d-1-i}$ for any $0 < i < d-1$ by Poincaré's duality theorem. (Statements of Sections 2 and 4 hold over any field $\mathbf{k}$.)

Definition 1.6: The h-vector of a $(d-1)$-dimensional simplicial complex Δ is a vector $h(\Delta) = (h_0, h_1, \dots, h_d)$ such that

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i},$$

or, equivalently,

$$f_{l-1}(\Delta) = \sum_{j=0}^{d} \binom{d-j}{d-l} h_j.$$

Remark: f-numbers of a $(d-1)$-dimensional homology manifold Δ satisfy certain linear relations, known as Dehn-Sommerville relations. These relations were derived by V. Klee [7] (for a more general class of simplicial complexes, namely, all simplicial complexes Δ satisfying $\chi(\mathrm{lk}\, F) = 1 + (-1)^{\dim \mathrm{lk} F}$ for any non-empty face F of Δ) (see also [10, 19]). These relations can be conveniently expressed in terms of h-numbers. They assert that

$$(1) \qquad h_{d-i} - h_i = (-1)^i \binom{d}{i} (\chi(\Delta) - (1 + (-1)^{d-1})) \quad \text{for } i = 0, 1, \ldots, d.$$

We will use several versions of these relations (see Lemmas 5.1, 7.3 below).

We prove the following result concerning face numbers and Betti numbers of homology manifolds.

THEOREM 1.7: *Let Δ be a $(d-1)$-dimensional Buchsbaum complex on n vertices. Let*

$$h'_j(\Delta) = h_j(\Delta) + \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \beta_{i-1}(\Delta) \quad \text{for } j = 0, 1, \ldots, d.$$

Then, $h'_0(\Delta) = 1$, $h'_1(\Delta) = n - d$ and

$$h'_{r+1}(\Delta) \leq \left(h'_r(\Delta) - \binom{d-1}{r} \beta_{r-1}(\Delta) \right)^{<r>} \quad \text{for } r = 1, 2, \ldots, d-1,$$

where, for

$$a = \binom{n_r}{r} + \binom{n_{r-1}}{r-1} + \cdots + \binom{n_i}{i}$$

(where $n_r > n_{r-1} > \cdots > n_i \geq i \geq 1$), we define

$$a^{<r>} = \binom{n_r + 1}{r+1} + \binom{n_{r-1}+1}{r} + \cdots + \binom{n_i + 1}{i+1}.$$

This theorem is similar in spirit to a theorem of Björner and Kalai on face numbers and Betti numbers of simplicial complexes. (But here we have only necessary conditions rather than a full characterization.)

Remark: The main reason for defining the modified h-vector, h', is the Schenzel theorem (see Theorem 2.7 below).

To describe our results for arbitrary homology manifolds we need the following definition:

Definition 1.8: A simplicial complex Δ is said to be l-neighborly complex if any l of its vertices form a face in Δ.

l-neighborliness has a simple interpretation in terms of h-numbers: a complex Δ is l-neighborly iff

$$h_i(\Delta) = \binom{n-d+i-1}{i} \quad \text{for any } i \leq l.$$

Note that all d-dimensional cyclic polytopes are $\lfloor d/2 \rfloor$-neighborly and it is known (and also follows from the UBT) that simplicial d-polytopes and, more generally, triangulations of $(d-1)$-dimensional spheres with more than $d+1$ vertices cannot be $(\lfloor d/2 \rfloor + 1)$-neighborly. The UBT for odd-dimensional homology manifolds implies that no $(2k-1)$-dimensional homology manifold with more than $2k+1$ vertices can be $(k+1)$-neighborly.

On the other hand, a $2k$-dimensional homology manifold Δ with $n > 2k+2$ vertices **may** be $(k+1)$-neighborly. If this is the case, then

$$f_i(\Delta) = \binom{n}{i+1} \quad \text{for } i = 0, \ldots, k.$$

It follows from the Dehn–Sommerville relations (see (1)) that for a $2k$-dimensional homology manifold Δ **all** face numbers of Δ are completely determined by $f_0(\Delta)$, $f_1(\Delta), \ldots, f_k(\Delta)$ as certain linear combinations. Substituting in these linear combinations

$$f_i = \binom{n}{k+1} \quad \text{for } i = 0, \ldots, k$$

we obtain an expression for the number of i-faces of any $(k+1)$-neighborly $2k$-dimensional homology manifold with n vertices (for any $1 \leq i \leq 2k$). Denote this number by $M_{i+1}(n, 2k+1)$. (For an explicit formula for $M_i(n, 2k+1)$ see Section 6. Note that the numbers $M_i(n, 2k+1)$ are determined formally from the linear relations; we define them even if a $(k+1)$-neighborly $2k$-dimensional homology manifold with n vertices does not exist. Moreover, note that the numbers $M_i(n, 2k+1)$ are rationals, and in general are not integers, so for some n and k, a $(k+1)$-neighborly $2k$-dimensional homology manifold cannot exist.)

THEOREM 1.9 (UBC′): *Let Δ be a $2k$-dimensional homology manifold on n vertices. Then*

$$f_{i-1}(\Delta) \leq M_i(n, 2k+1) \quad \text{for } i = 2, \ldots, 2k+1.$$

If equalities are attained in all these inequalities, then Δ *is* $(k+1)$*-neighborly,* $\beta_i(\Delta) = 0$ *unless* $i = k$ *or* $i = d$*, and*

$$\beta_k = \binom{n-k-2}{k+1} \Big/ \binom{2k+1}{k}.$$

Kühnel conjectured ([9]) that for any $2k$-dimensional combinatorial manifold Δ with n vertices

$$(-1)^k(\chi(\Delta) - 2) \leq \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}$$

and equality holds iff Δ is $(k+1)$-neighborly. This conjecture was known to be true for $k \leq 2$ and also for $n \leq 3k+3$ and $n \geq k^2+4k+3$ (see [10]). In Section 5 we give a proof of the Kühnel conjecture for all $2k$-dimensional homology manifolds with n vertices, where $n \leq 3k+3$ or $n \geq 4k+3$. We prove, in fact, that for such n Kühnel's upper bound applies even for $\beta_k + 2(\beta_{k-2} + \beta_{k-3} + \cdots + \beta_1 + \beta_0)$. We also prove that for $n \leq 3(k+1)$ or $n \geq 7k+4$ Kühnel's upper bound applies even for $\sum_{i=1}^{2k-1} \beta_i$. In addition, we find a similar upper bound on the sum of the Betti numbers for odd-dimensional homology manifolds.

Equality in Theorem 1.9 and the Kühnel conjecture holds if Δ is a $(k+1)$-neighborly $2k$-dimensional homology manifold. The existence of a $(k+1)$-neighborly triangulation of a $2k$-dimensional (topological) manifold for $k = 1$ and infinitely many n is the famous Heawood conjecture settled by Ringel and Youngs [14]. There are only five known examples for higher values of k (see [10]), but it is plausible that for every k there are infinitely many examples.

The structure of the paper is as follows. Section 2 contains background material on Buchsbaum complexes. The proofs of our main results are given in Sections 3–6. In Section 7 we discuss several further conjectures. The proofs are based on studying the combinatorics of the "shifted model" [2] (i.e. the generic initial ideal [4]) of the Stanley–Reisner ring of Buchsbaum complexes.

2. Buchsbaum complexes

In this section we review some facts from commutative algebra and topology that produce an inequality for the h-vector of a Buchsbaum complex.

Definition 2.1: Suppose that R is a finitely generated standard graded algebra over a field $\mathbf{k}$. That is, $R = \bigoplus_{i=0}^{\infty} R_i$, where $R_0 \cong \mathbf{k}$ and R_i is a finite-dimensional vector space over $\mathbf{k}$, such that $R_i R_j = R_{i+j}$ for all $i, j \in \mathbf{N}$. (Elements of R_i are called i-homogeneous elements of R.) Then

- $H(R,i) = \dim_{\mathbf{k}} R_i$ is called the Hilbert function of R;
- $F(R,x) = \sum_i H(R,i)x^i$ is called the Poincaré series of R.

Definition 2.2: Let R be a finitely generated graded algebra over $\mathbf{k}$.

- The Krull dimension of R ($\dim R$) is the maximum number of algebraically independent (over $\mathbf{k}$) elements $\theta_1, \ldots, \theta_d$ in $\bigoplus_{i=1}^{\infty} R_i$.
- A sequence $\theta_1, \ldots, \theta_d \in \bigcup_{i=1}^{\infty} R_i$ is called an h.s.o.p. (homogeneous system of parameters) for R iff $d = \dim R$ and $\dim R/(\theta_1, \ldots, \theta_d) = 0$. (Equivalently, $\theta_1, \ldots, \theta_d \in \bigcup_{i=1}^{\infty} R_i$ is an h.s.o.p for R iff $d = \dim R$ and R is a finitely-generated $\mathbf{k}[\theta_1, \ldots, \theta_d]$-module.)

Let R be a standard graded finitely generated algebra over field $\mathbf{k}$. It is well-known (see, for example, [1]) that

- $F(R,x) = P_R(x)/(1-x)^d$, where $d = \dim R, P_R(x) \in \mathbf{Z}[x], P_R(0) = 1, P_R(1) \neq 0$.
- For sufficiently large m,

$$H(R,m) \in \mathbf{Q}[m], \quad \text{where } \deg(H(R,m)) = d-1.$$

Definition 2.3: Let Δ be a simplicial complex on the set of vertices $V = \{x_1, x_2, \ldots, x_n\}$. The Stanley Reisner ring of Δ is $R_\Delta = \mathbf{k}[x_1, \ldots, x_n]/I_\Delta$, where I_Δ is the ideal in $\mathbf{k}[x_1, \ldots, x_n]$, generated by all square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_s}$ such that $\{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\} \notin \Delta$.

Define $\deg x_i = 1$ for $i = 1, 2, \ldots, n$. This makes R_Δ into a graded ring.

CLAIM 2.4 (Stanley [18], [19]): *For a simplicial complex Δ*

$$H(R_\Delta, m) = \begin{cases} 1 & \text{if } m = 0, \\ \sum_{i=0}^{\dim\Delta} f_i(\Delta) \cdot \binom{m-1}{i} & \text{if } m > 0. \end{cases}$$

In particular,

$$\dim R_\Delta = 1 + \dim \Delta.$$

COROLLARY 2.5: *If Δ is a simplicial complex of dimension δ, then*

$$(1-x)^{1+\delta} F(R_\Delta, x) = \sum_{l=0}^{\delta+1} h_l x^l. \tag{2}$$

Definition 2.6: A finite connected simplicial complex Δ is called a Buchsbaum complex (over $\mathbf{k}$) if for any $p \in X = |\Delta|$ and any $i < \dim \Delta$, $H_i(X, X-p; \mathbf{k}) = 0$, where $H_i(X, X-p; \mathbf{k})$ is an i-th relative singular homology over $\mathbf{k}$.

For example, all connected homology manifolds are Buchsbaum complexes over any field (this follows immediately from the definition of a homology manifold).

THEOREM 2.7 (Schenzel [16]): *Let Δ be a $(d-1)$-dimensional Buchsbaum complex and let $\beta_{-1}, \beta_0, \ldots, \beta_{d-1}$ be its reduced Betti numbers. If $\theta_1, \ldots, \theta_d \in (R_\Delta)_1$ is an h.s.o.p. for R_Δ, then*

$$(3)\quad (1-x)^d F(R_\Delta, x) = F(R_\Delta/(\theta_1, \ldots, \theta_d), x) + \sum_{j=1}^{d} \binom{d}{j} \left(\sum_{i=0}^{j-1} (-1)^{j-i} \beta_{i-1} \right) x^j.$$

Define $(h'_0, h'_1, \ldots, h'_d)$ by

$$\sum_{j=0}^{d} h'_j x^j = F(R_\Delta/(\theta_1, \ldots, \theta_d), x).$$

Then, by Corollary 2.5, we can rewrite (3) as

$$(4)\qquad h_j = h'_j - \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \beta_{i-1}, \quad \text{for } j = 0, 1, \ldots, d.$$

Now, we fix an infinite field $\mathbf{k}$. Suppose that Δ is a $(d-1)$-dimensional complex on n vertices. In particular, $\dim R_\Delta = d$. By the Noether Normalization Lemma there exists an h.s.o.p. for R_Δ. Moreover, since $\mathbf{k}$ is infinite and R_Δ is generated by 1-homogeneous elements, we can choose an h.s.o.p. $\theta_1, \ldots, \theta_d$ from $(R_\Delta)_1$. Let $\theta_1, \ldots, \theta_d$ be such an h.s.o.p. Denote by S_i the i-th homogeneous component of $S = R_\Delta/(\theta_1, \ldots, \theta_d)$. Now, observe that

$$(5)\qquad h'_1 = \dim_{\mathbf{k}} S_1 = \dim_{\mathbf{k}} (R_\Delta)_1 - \dim_{\mathbf{k}}(\mathrm{Span}\{\theta_1, \ldots, \theta_d\}) = n - d$$

and $S = R_\Delta/(\theta_1, \ldots, \theta_d)$ is generated by S_1. Therefore, we obtain that the dimension of S_i (over $\mathbf{k}$) is not bigger than the total number of monomials of degree i in $n-d$ variables. Therefore,

$$(6)\qquad h'_i = \dim_{\mathbf{k}} S_i \leq \binom{n-d+i-1}{i}.$$

Combining (4) and (6), we observe that for a $(d-1)$-dimensional Buchsbaum complex Δ on n vertices

$$(7)\quad h_j(\Delta) \leq \binom{n-d+j-1}{j} - \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \beta_{i-1}(\Delta) \quad \text{for } j = 0, 1, \ldots, d.$$

Remark: In Section 4 we will obtain much stronger inequalities than (6) (see Theorem 1.7). In particular, from that theorem it follows that if $\beta_i(\Delta) > 0$ for some $0 < i < d-1$ then

$$h'_j < \binom{n-d+j-1}{j} \quad \text{for all } j \geq i+2.$$

More precisely, it follows that

$$h'_j = \binom{n-d+j-1}{j} \quad \text{for some } j$$

implies that

$$h'_i = \binom{n-d+i-1}{i} \quad \text{for all } i < j$$

and $\beta_i = 0$ for all $i < j-1$; then by (4),

$$h_i = h'_i = \binom{n-d+i-1}{i} \quad \text{for all } i \leq j,$$

and so Δ is j-neighborly.

3. The proof of the UBC for odd-dimensional homology manifolds

In this section we prove the UBC for odd-dimensional homology manifolds. The proof is very simple; it follows at once from the Schenzel theorem and Dehn–Sommerville relations.

LEMMA 3.1: *For natural d, i, m*

$$\sum_{k=0}^{m}(-1)^k\binom{d}{i+k} = \binom{d-1}{i-1} + (-1)^m\binom{d-1}{i+m}.$$

Proof: Follows immediately from the fact that for any j

$$\binom{d}{j} = \binom{d-1}{j-1} + \binom{d-1}{j}. \quad \blacksquare$$

Now we are ready to prove Theorem 1.4. Let Δ be a $(2k-1)$-dimensional homology manifold with n vertices. From Poincaré's duality theorem, it follows that $\chi(\Delta) = 0$, and so by Dehn–Sommerville relations (1)

$$h_i(\Delta) = h_{2k-i}(\Delta) \quad \text{for } i = 0, 1, \ldots, 2k.$$

Using this and the definition of h-numbers, we obtain that

$$f_{l-1}(\Delta)=\sum_{j=0}^{k}{}^{*}\left(\binom{2k-j}{2k-l}+\binom{j}{2k-l}\right)h_j(\Delta) \tag{8}$$

for $l=1,\dots,2k$, where we used the following notation:

$$\sum_{j=m}^{k}{}^{*}\phi(j):=\sum_{j=m}^{k-1}\phi(j)+\frac{1}{2}\phi(k) \quad \text{for any } \phi \text{ and } m\le k.$$

Thus, for $l=1,\dots,2k$,

$$\begin{aligned}
f_{l-1}(\Delta) &\overset{\text{by (8)}}{=} \sum_{j=0}^{k}{}^{*}\left(\binom{2k-j}{2k-l}+\binom{j}{2k-l}\right)h_j(\Delta)\\
&\overset{\text{by (7)}}{\le} \sum_{j=0}^{k}{}^{*}\left(\binom{2k-j}{2k-l}+\binom{j}{2k-l}\right)\\
&\qquad\left[\binom{n-2k+j-1}{j}-\binom{2k}{j}\sum_{i=0}^{j-1}(-1)^{j-i-1}\beta_{i-1}\right]\\
&=\sum_{j=0}^{k}{}^{*}\left(\binom{2k-j}{2k-l}+\binom{j}{2k-l}\right)h_j(C_{2k}(n))\\
&\quad-\sum_{i=0}^{k-1}\left[\sum_{j=i+1}^{k}{}^{*}(-1)^{j-i-1}\binom{2k}{j}\left(\binom{2k-j}{2k-l}+\binom{j}{2k-l}\right)\right]\beta_{i-1}\\
&\overset{\text{by (8)}}{=} f_{l-1}(C_{2k}(n))\\
&\qquad-\sum_{i=0}^{k-1}\left[\sum_{j=i+1}^{k}{}^{*}(-1)^{j-i-1}\binom{2k}{j}\left(\binom{2k-j}{2k-l}+\binom{j}{2k-l}\right)\right]\beta_{i-1}\\
&=f_{l-1}(C_{2k}(n))-\sum_{i=0}^{k-1}\left[\sum_{j=i+1}^{2k-i-1}(-1)^{j-i-1}\binom{2k}{j}\binom{j}{2k-l}\right]\beta_{i-1}\\
&\overset{\binom{2k}{j}\binom{j}{2k-l}=\binom{2k}{l}\binom{l}{2k-j}}{=} f_{l-1}(C_{2k}(n))\\
&\qquad-\sum_{i=0}^{k-1}\binom{2k}{l}\left[\sum_{j=i+1}^{2k-i-1}(-1)^{j-i-1}\binom{l}{2k-j}\right]\beta_{i-1}\\
&\le f_{l-1}(C_{2k}(n)),
\end{aligned}$$

since by Lemma 3.1

$$\sum_{j=i+1}^{2k-i-1}(-1)^{j-i-1}\binom{l}{2k-j}=\binom{l-1}{i}+\binom{l-1}{2k-i-1}\geq 0 \text{ for } i=0,1,\ldots,k-1.$$

Remark: The proof of Theorem 1.4 gives upper bounds on $f_l(\Delta)$ in terms of $f_l(C_{2k}(n))$ and Betti numbers of Δ. Namely, we obtain

$$f_l(\Delta)\leq f_l(C_{2k}(n))-\sum_{i=0}^{k-1}\binom{2k}{l+1}\left[\binom{l}{i}+\binom{l}{2k-i-1}\right]\beta_{i-1}. \tag{9}$$

However, since the upper bounds on h-numbers that we used, (7), are not sharp (see remark at the end of Section 2), these upper bounds are not sharp as well in the following sense: if $\beta_i(\Delta)\neq 0$ for some $i<k-1$ then these inequalities are strict for all $l\geq i+2$.

The other consequence of (9) is that a $(2k-1)$-dimensional homology manifold can be k-neighborly only if all $\beta_i(\Delta)=0$ except β_{2k-1} and, possibly, $\beta_{k-1}=\beta_k$ (compare Theorem 1.9). (It is known that for any (topological) 3-manifold there is a 2-neighborly triangulation if the number of vertices is chosen to be sufficiently large, see [21], [15]; however, it is not known whether all $(k-2)$-connected $(2k-1)$-manifolds admit a k-neighborly triangulation for $k>2$.)

4. New inequalities

In this section, using facts from commutative algebra for local cohomology of Buchsbaum modules and the shifting argument applied to the Stanley–Reisner ring, we provide new relations between sequences $\{h'(\Delta)_j\}$ and $\{h'_j(\Delta)-\binom{d-1}{j}\beta_{j-1}(\Delta)\}$, where Δ is a $(d-1)$-dimensional Buchsbaum complex. These relations together with Macaulay's theorem give new inequalities (see Theorem 1.7), much stronger than (6).

These inequalities were conjectured by G. Kalai. He also suggested a way for their proof.

Let Δ be a $(d-1)$-dimensional Buchsbaum complex on n vertices. Let $R_\Delta=\mathbf{k}[x_1,\ldots,x_n]/I_\Delta$ be the Stanley–Reisner ring of Δ. Let $y_i=\sum_{j=1}^{n}\lambda_i^j x_j$ for $i=1,2,\ldots,n$ be generic combinations of $x_1,\ldots,x_n$ (see [2]). For $r=0,1,\ldots$ denote by M_r the set of all monomials in $y_1,\ldots,y_n$ of degree r.

Now, we choose a basis of R_Δ over $\mathbf{k}$ in the following way:

1. For $r=1,2,\ldots$ we order the elements of M_r in the lexicographic order ($<_{lex}$). Lexicographic order is a linear order, so we can write

$$M_r=\{m_1^r<_{lex}m_2^r<_{lex}m_3^r<_{lex}\cdots\}.$$

2. We define $S(\Delta) = \bigcup_{r=1}^{\infty} S_r(\Delta)$, where

$$S_r(\Delta) = \{m_i^r \in M_r : m_i^r \text{ is linearly independent of } m_1^r, \ldots, m_{i-1}^r \text{ in } R_\Delta\}.$$

It follows immediately from this construction that $S(\Delta)$ is a basis of R_Δ over $\mathbf{k}$, and that $S(\Delta)$ is an order ideal of monomials. (That is, if $m \in S(\Delta)$ and m' is a divider of m, then m' is also in $S(\Delta)$.) Moreover, $S(\Delta)$ is shifted (see [2]). (That is, if $m = x_{i1}x_{i2}\cdots x_{ir}$ is in $S(\Delta)$, where $i_1 \leq i_2 \leq \cdots \leq i_r$, and there are some numbers $1 \leq j_1, j_2, \ldots, j_r$ such that $j_1 \leq i_1, j_2 \leq i_2, \ldots, j_r \leq i_r$, then $x_{j_1}x_{j_2}\cdots x_{j_r}$ is also in $S(\Delta)$.)

Let $\mathrm{MON}(i) := \{m : m \text{ is a monomial in } y_i, y_{i+1}, \ldots, y_n\}$. For $i = 0, 1, \ldots, d-1$ define

$$A_i = \{m \in S(\Delta) \cap \mathrm{MON}(i+1) : m \notin \bigcup_{j=0}^{i-1} A_j \text{ and } \exists k \in \mathbf{N} \text{ s.t. } y_{i+1}^k \cdot m \notin S(\Delta)\}.$$

In other words, (since $S(\Delta)$ is shifted) $m \in A_i$ iff m satisfies the following conditions:

- m is a monomial in the variables $y_{i+1}, \ldots, y_n$;
- m belongs to $S(\Delta)$ and, moreover, all monomials obtained from m by multiplying it by any monomial in the variables $y_1, \ldots, y_i$ are in $S(\Delta)$, but there is a monomial n in the variables $y_1, \ldots, y_{i+1}$ such that $nm \notin S(\Delta)$.

Define

$$\begin{aligned} A_d &= (S(\Delta) \cap \mathrm{MON}(d+1)) - \bigcup_{i=1}^{d-1} A_i \\ &= \{m \in \mathrm{MON}(d+1) : \forall k \in \mathbf{N}\ y_d^k \cdot m \in S(\Delta)\}. \end{aligned}$$

In other words, (since $S(\Delta)$ is shifted) A_d is the set of all monomials m in the variables $y_{d+1}, \ldots, y_n$ such that $m \in S(\Delta)$ and the product of m and any monomial in the variables $y_1, \ldots, y_d$ belongs to $S(\Delta)$.

Let $P_i(x)$ be the generating function of A_i (that is, $P_i(x) = \sum_{j \geq 0} l_i^j x^j$, where l_i^j is the number of monomials in A_i of degree j). It follows easily from the definition of $A_0, A_1, \ldots, A_d$ (and the fact that $S(\Delta)$ is shifted) that any monomial $m \in S(\Delta)$ can be written uniquely as $m = n'm'$, where $m' \in A_i$ for some $0 \leq i \leq d$ and n' is a monomial in $y_1, \ldots, y_i$. Since $S(\Delta)$ is a basis of R_Δ over $\mathbf{k}$ this implies that

$$R_\Delta = \bigoplus_{i=0}^{d} \left(\bigoplus_{\eta \in A_i} \eta \cdot \mathbf{k}[y_1, \ldots, y_i] \right).$$

Thus,

$$\frac{\sum_{j=0}^{d} h_j(\Delta)x^j}{(1-x)^d} \overset{\text{Cor. 2.5}}{=} F(R_\Delta, x) = \sum_{i=0}^{d}\left(\sum_{\eta\in A_i} x^{\deg \eta} F(\mathbf{k}[y_1,\ldots,y_i],x)\right)$$
$$= \sum_{i=0}^{d}\sum_{\eta\in A_i}\frac{x^{\deg\eta}}{(1-x)^i} = \sum_{i=0}^{d}\frac{P_i(x)}{(1-x)^i}.$$

In other words,

$$P_d(x) = h(x) - \sum_{i=0}^{d-1} P_i(x)(1-x)^{d-i}, \tag{10}$$

where $h(x) = \sum_{i=0}^{d} h_i x^i$.

Using facts for local cohomology of Buchsbaum modules, we will prove

LEMMA 4.1:

$$P_m(x) = \sum_{j=0}^{m-1}\binom{m-1}{j}\beta_j x^{j+1} \quad \textit{for } m = 1,2,\ldots,d-1;$$

$P_0 = 0$ *and, therefore,* $A_0 = \emptyset$.

We postpone the proof of this lemma until the end of the section.

From (10) and Lemma 4.1 we obtain

$$P_d(x) = h(x) - \sum_{m=1}^{d-1}\left[\sum_{i=0}^{m-1}\binom{m-1}{i}\beta_i x^{i+1}\right](1-x)^{d-m}.$$

Let $P_d(x) = \sum_{j=0}^{d} k_j x^j$. Then

$$\begin{aligned} k_j &= h_j - \sum_{m=1}^{d-1}\sum_{i=0}^{j-1}\beta_i\binom{m-1}{i}(-1)^{j-i-1}\binom{d-m}{j-i-1} \\ &= h_j - \sum_{i=0}^{j-1}\left[(-1)^{j-i-1}\sum_{m=1}^{d-1}\binom{m-1}{i}\binom{d-m}{j-i-1}\right]\beta_i. \end{aligned} \tag{11}$$

If $i = j-1$, then

$$\sum_{m=1}^{d-1}\binom{m-1}{i}\binom{d-m}{j-i-1} = \sum_{m=j}^{d-1}\binom{m-1}{j-1} = \sum_{m=j-1}^{d-2}\binom{m}{j-1} = \binom{d-1}{j}. \tag{12}$$

If $i < j-1$, then

$$\sum_{m=1}^{d-1}\binom{m-1}{i}\binom{d-m}{j-i-1} = \sum_{m=i+1}^{d-j+i+1}\binom{m-1}{i}\binom{d-m}{j-i-1} = \binom{d}{j}. \tag{13}$$

From (11)-(13) it follows that

$$\begin{aligned} k_j &= h_j - \binom{d-1}{j}\beta_{j-1} + \binom{d}{j}\sum_{i=0}^{j-1}(-1)^{j-i-1}\beta_{i-1} \\ &\overset{\text{by (4)}}{=} h'_j - \binom{d-1}{j}\beta_{j-1}. \end{aligned}$$

Thus, we obtain

LEMMA 4.2:

$$P_d(x) = \sum_{j=0}^{d}\left(h'_j - \binom{d-1}{j}\beta_{j-1}\right)x^j. \quad \blacksquare$$

LEMMA 4.3: $\sum_{m=0}^{d} P_m(x) = \sum_{j=0}^{d} h'_j x^j$.

Proof: By Lemmas 4.1 and 4.2

$$\begin{aligned} \sum_{m=0}^{d} P_m(x) &= \sum_{m=1}^{d-1}\sum_{j=1}^{m}\binom{m-1}{j-1}\beta_{j-1}x^j + \sum_{j=0}^{d}\left(h'_j - \binom{d-1}{j}\beta_{j-1}\right)x^j \\ &= \sum_{j=0}^{d}\left(h'_j - \left(\binom{d-1}{j} - \sum_{m=1}^{d-1}\binom{m-1}{j-1}\right)\beta_{j-1}\right)x^j = \sum_{j=0}^{d} h'_j x^j. \quad \blacksquare \end{aligned}$$

Let $\tilde{S}(\Delta) := S(\Delta) \cap \mathrm{MON}(d+1)$ and $\tilde{S}_r(\Delta) := \{m \in \tilde{S}(\Delta) : \deg m = r\}$. So, $\tilde{S}(\Delta)$ is a basis of $R_\Delta/(y_1, \ldots, y_d)$ over $\mathbf{k}$. Now, note that $y_1, \ldots, y_d$ is an h.s.o.p. for R_Δ. Therefore, by the definition of h' (see Section 2), $h'(x) = \sum_{j=0}^{d} h'_j x^j$ is a generating function of $\tilde{S}(\Delta)$. Since P_i is a generating function of A_i for $i = 0, \ldots, d$ and $\bigcup_{i=1}^{d} A_i \supset \tilde{S}(\Delta)$, we obtain from Lemma 4.3

COROLLARY 4.4: $\bigcup_{i=1}^{d} A_i = \tilde{S}(\Delta)$. *In particular, for* $i = 1, \ldots, d$, $A_i \subset \tilde{S}(\Delta)$, *and so* $A_i \subset \mathrm{MON}(d+1)$.

This corollary implies

COROLLARY 4.5: *For any* $i = 1, \ldots, d-1$ *and any* $m \in A_i$, $y_{i+1} \cdot m \notin S(\Delta)$.

Proof: Suppose that there is $m \in \mathrm{MON}(d+1)$ such that $y_{i+1} \cdot m \in S(\Delta)$. Note that $y_{i+1} \cdot m \notin \mathrm{MON}(d+1)$. So by Corollary 4.4, $y_{i+1} \cdot m \notin \bigcup_{j=1}^{i} A_j$. Therefore, by the definition of A_i, for all $k \in \mathbf{N}$, $y_{i+1}^k \cdot y_{i+1} m = y_{i+1}^{k+1} \cdot m \in S(\Delta)$. Thus, $m \notin A_i$. $\blacksquare$

COROLLARY 4.6: *The shadow of the set $\tilde{S}_{r+1}(\Delta)$, $\partial_{r+1}\tilde{S}_{r+1}(\Delta)$, is contained in A_d (where, for a monomial m of degree l, we define the shadow of m, $\partial_l\{m\}$, by $\partial_l\{m\} = \{m' : \deg m' = l-1, \text{ and } m'|m\}$, and for a set B of monomials of degree l, we define the shadow of B, $\partial_l B$, by $\partial_l B = \bigcup_{m\in B}\partial_l\{m\}$).*

Proof: $S(\Delta)$ is an order ideal, so by definition of $\tilde{S}(\Delta)$, $\tilde{S}(\Delta)$ is also an order ideal. Thus, $\partial_{r+1}\tilde{S}_{r+1}(\Delta) \subset \tilde{S}_r(\Delta)$. By definition of A_d, $\tilde{S}_r(\Delta) - A_d = (A_1 \cup A_2 \cup \cdots \cup A_{d-1}) \cap \tilde{S}_r(\Delta)$, so, by Corollary 4.5, for each $m \in \tilde{S}_r(\Delta) - A_d$ there is an $i = i(m) \le d$, s.t. $y_i \cdot m \notin S(\Delta)$. Therefore, $y_{d+1}\cdot m$, $y_{d+2}\cdot m, \ldots y_n \cdot m \notin S(\Delta)$, since $S(\Delta)$ is shifted. Thus, $m \notin \partial_{r+1}\tilde{S}_{r+1}(\Delta)$. Therefore, $\partial_{r+1}\tilde{S}_{r+1} \cap (\tilde{S}_r - A_d) = \emptyset$, and so $\partial_{r+1}\tilde{S}_{r+1} \subset \tilde{S}_r \cap A_d$. ∎

Since $P_d(x) = \sum(h'_j - \binom{d-1}{j}\beta_{j-1})x^j$ is a generating function of A_d and $h'(x) = \sum_{j=0}^{d} h'_j x^j$ is a generating function of $\tilde{S}$, Corollary 4.6, (4) and Macaulay's theorem (see [5]) imply Theorem 1.7, which asserts that for a $(d-1)$-dimensional Buchsbaum complex Δ on n vertices, $h'_0 = 1$, $h'_1 = n-d$ and

$$h'_{r+1} \le \left(h'_r - \binom{d-1}{r}\beta_{r-1}\right)^{<r>} \qquad \text{for } r = 1, 2, \ldots, d-1.$$

In fact, nowhere in this paper do we need an exact expression for $\left(h'_r - \binom{d-1}{r}\beta_{r-1}\right)^{<r>}$, which Macaulay's theorem provides. Rather, it will be sufficient for our purposes to use some upper bounds for $\left(h'_r - \binom{d-1}{r}\beta_{r-1}\right)^{<r>}$

Now we are going to prove Lemma 4.1. First, we review some relevant commutative algebra. Let $M = \bigoplus_{i\in\mathbf{Z}} M_i$ be a finitely generated module of dimension d over a finitely generated graded ring $R = \bigoplus_{j=0}^{\infty} R_j$. (That is, $R_j \cdot M_i \subset M_{i+j}$ for all i and j. The (Krull) dimension of M, $\dim M$, is defined as a dimension of the ring $R/\mathrm{Ann}M$.) Let $m = \bigoplus_{j=1}^{\infty} R_i$ be the irrelevant ideal of R.

Definition 4.7: A system of elements $x_1, \ldots, x_r \in m$ is called a weak M-sequence if for each $i = 1, \ldots, r$

$$(x_1, \ldots, x_{i-1})M : x_i = (x_1, \ldots, x_{i-1})M : m,$$

where for an ideal α of R and a submodule N of M $N : \alpha = \{u \in M : \alpha\cdot u \subset N\}$. In particular, for $i = 1$ we have $0 : x_1 = 0 : m$.

Definition 4.8: A family $(x_1, \ldots, x_d)$ of elements $x_1, \ldots, x_d$ of m is said to be a system of parameters of M if $\dim M/(x_1, \ldots, x_d)M = 0$.

(By the Noether Normalization Lemma a system of parameters allways exists.)

Definition 4.9: M is called a Buchsbaum module if every system of parameters of M is a weak M-sequence. R is called a Buchsbaum ring if it is a Buchsbaum module as a module over itself.

It is known (see [16]) that for a finite simplicial complex Δ, Δ is a Buchsbaum complex in the sense of Section 2 iff R_Δ is a Buchsbaum ring.

Denote by $H^i(M)$ the local cohomology module of M. We recall that $H^i(M)$ is a graded module and that

$$H^0(M) = \{u \in M : m^k \cdot u = 0 \text{ for some } k\}.$$

Now, suppose that M is a Buchsbaum module. Denote by $F_i(x) = F(H^i(M), x)$ the Poincaré series of $H^i(M)$. For $i = 0, 1, \ldots, 2(d-1)$ we define modules M_i as follows: $M_0 = M$, $M_1 = M_0/H^0(M_0)$. Suppose that M_{2j-1} is already defined. Let $x_j \in m$ be a non-zero divisor on M_{2j-1} of degree one. Define $M_{2j} := M_{2j-1}/x_j M_{2j-1}$ and $M_{2j+1} = M_{2j}/H^0(M_{2j})$. It is a known fact from the commutative algebra (as proved by D. Eisenbud and R. Stanley, see Appendix) that for a Buchsbaum complex M

$$(14) \quad F(H^0(M_{2k}), x) = \sum_{j=0}^{k-1} \binom{k-1}{j} F_{j+1}(x) x^{j+1} \quad \text{for } k = 1, 2, \ldots, d-1,$$

where $F(H^0(M_{2k}), x)$ is a Poincare series of $H^0(M_{2k})$.

Proof of Lemma 4.1: Let $R = M = R_\Delta$ and $M_{2i} := M_{2i-1}/y_i M_{2i-1}$. (More precisely: $M_{2i} = M_{2i-1}/\bar{y}_i M_{2i-1}$, where $\bar{y}_i$ is the image of y_i in M_{2i-1} under the natural homomorphism.) Then by (14)

$$F(H^0(M_{2k}), x) = \sum_{j=0}^{k-1} \binom{k-1}{j} F_{j+1}(x) x^{j+1} \quad \text{for } k = 1, 2, \ldots, d-1.$$

Since Δ is a Buchsbaum complex, $H^i(M) = H^i(R_\Delta) \cong \tilde{H}_{i-1}(\Delta)$ for $i \leq d-1$ (see Corollary 4.13 on page 144 of [20]) and all elements of the module $H^i(M) = H^i(R_\Delta)$ have degree 0 (see Lemma 2.5 on page 117 of [20]). So, actually, F_i is a number, rather than a formal series, and for $j \leq d-2$

$$F_{j+1} = F(H^{j+1}(M), x) = \dim_{\mathbf{k}}(H^{j+1}(M)) = \dim_{\mathbf{k}}(\tilde{H}_j(\Delta) = \beta_j(\Delta).$$

Thus,

$$F(H^0(M_{2k}), x) = \sum_{j=0}^{k-1} \binom{k-1}{j} \beta_j x^{j+1} \quad \text{for } k = 1, \ldots, d-1.$$

Also, since $M_0 = R_\Delta$, $H^0(M_0) = 0$. Therefore, to complete the proof of the Lemma, it is sufficient to show that the generating function of $H^0(M_{2k})$ and that of A_k coincide for any $k = 0, 1, \dots, d-1$. We will prove a slightly stronger result:

Let $B_k = \mathrm{MON}(k+1) \cap S(\Delta) - \bigcup_{j=0}^{k} A_j$ for $k = 0, 1, \dots, d-1$, so B_k is the set of all monomials u in $\mathrm{MON}(k+1) \cap S(\Delta)$, such that for any $l \in \mathbf{N}$ and $j \leq k+1$, $x_j^l \cdot u \in S(\Delta)$. We prove by induction that

1. $B_k \cup A_k$ (more precisely, the image of $B_k \cup A_k$ in M_{2k}) is a basis of M_{2k} over $\mathbf{k}$;
2. the generating functions of $H^0(M_{2k})$ and A_k are equal;
3. B_k is a basis of M_{2k+1} over $\mathbf{k}$.

Suppose that the assertion is true for $k-1$, so B_{k-1} is a basis of M_{2k-1}. Since $M_{2k} = M_{2k-1}/y_k M_{2k-1}$, this implies that $B_{k-1} \cap \mathrm{MON}(k+1) = B_k \cup A_k$ is a basis of M_{2k}. Let N_{2k} be the submodule of M_{2k} generated by B_k. From the definition of B_k it follows that for any l, $y_{k+1}^l \cdot B_k \subset B_k$. Since B_k is a basis of N_{2k}, $y_{k+1} \in m$ and $H^0(M_{2k}) = \{u \in M_{2k} : m^l \cdot u = 0 \text{ for some } l\}$, it follows that $N_{2k} \cap H^0(M_{2k}) = \{0\}$.

Since M is a Buchsbaum module and $(y_1, \dots, y_d)$ is a system of parameters of M, we have that $(y_1, \dots, y_d)$ is a weak M-sequence of M, so

$$\{u \in M_{2k} : m \cdot u = 0\} = \{u \in M_{2k} : y_{k+1} \cdot u = 0\}.$$

Therefore,

$$\begin{aligned} H^0(M_{2k}) &= \{u \in M_{2k} : m^l \cdot u = 0 \text{ for some } l\} \\ &= \{u \in M_{2k} : y_{k+1}^l \cdot u = 0 \text{ for some } l\} = \bigcup_{l \in \mathbf{N}} \mathrm{Ker}(y_{k+1}^l), \end{aligned} \tag{15}$$

where $y_{k+1}^l : u \mapsto y_{k+1}^l \cdot u$.

Since $B_k \cup A_k$ is a basis of M_{2k} and since it is shifted, we obtain from (15) and the definitions of A_k and B_k that the generating functions of $H^0(M_{2k})$ and A_k are equal. Therefore, for each $l \in \mathbf{N}$, $[N_{2k}]_l$ and $[H^0(M_{2k})]_l$ have complementary dimensions in $[M_{2k}]_l$ (here $[\ ]_l$ is an l-homogeneous part of the module). Since $N_{2k} \cap H^0(M_{2k}) = \{0\}$, we obtain that $M_{2k} = N_{2k} \oplus H^0(M_{2k})$. Recalling that B_k is a basis of N_{2k} we see that the image of B_k is a basis of $M_{2k+1} = M_{2k}/H^0(M_{2k})$. This completes the induction argument. It remains to note that for $k = 0$, $B_0 \cup A_0 = S(\Delta)$ is, indeed, a basis of $M_0 = R_\Delta$. ∎

5. Upper bounds on Betti numbers; Kühnel's conjecture

In this section, after proving some auxiliary results about h'-numbers and Betti numbers, we obtain upper bounds on (weighted) sum of the Betti numbers of homology manifolds (in both odd-dimensional and even-dimensional cases). In particular, we prove the Kühnel conjecture for arbitrary $2k$-dimensional homology manifold on n vertices where $n \geq 4k+3$ or $n \leq 3k+3$, and, in fact, some stronger results.

From now on, let Δ be a $(d-1)$-dimensional homology manifold on n vertices. We begin by obtaining another version of the Dehn–Sommerville relations (1) for homology manifolds.

LEMMA 5.1:

$$h'_{d-j}(\Delta) = h'_j(\Delta) + \binom{d}{j}(\beta_j(\Delta) - \beta_{j-1}(\Delta)) \quad \text{for } j = 0, 1, \ldots, d-1.$$

Proof: By the Dehn–Sommerville relations

$$(16)\quad h_{d-j} - h_j = (-1)^j \binom{d}{j} \left(\chi(\Delta) - (1 + (-1)^{d-1})\right) = (-1)^j \binom{d}{j} \sum_{i=0}^{d-2} (-1)^i \beta_i$$

for $j = 0, 1, \ldots, d$.

Then, by the Schenzel theorem (see (4)),

$$\begin{aligned} h'_{d-j} - h'_j &= \left(h_{d-j} + \binom{d}{j}(\beta_{d-j-2} - \beta_{d-j-3} + \beta_{d-j-4} - \ldots)\right) \\ &\quad - \left(h_j + \binom{d}{j}(\beta_{j-2} - \beta_{j-3} + \beta_{j-4} - \ldots)\right) \\ &\overset{\text{by Poincaré duality}}{=} (h_{d-j} - h_j) \\ &\quad + \binom{d}{j}\left((\beta_{j+1} - \beta_{j+2} + \beta_{j+3} - \ldots)\right. \\ &\quad \left. -(\beta_{j-2} - \beta_{j-3} + \beta_{j-4} - \ldots)\right) \\ &\overset{\text{by (16)}}{=} (-1)^j \binom{d}{j} \sum_{i=0}^{d-2} (-1)^i \beta_i \\ &\quad + \binom{d}{j}\left(\sum_{i=0}^{d-2} (-1)^{i-j-1} \beta_i + (\beta_j - \beta_{j-1})\right) \\ &= \binom{d}{j}(\beta_j - \beta_{j-1}). \quad \blacksquare \end{aligned}$$

Our second step is to obtain a weaker version of Theorem 1.7, which will be more suitable for the proofs.

LEMMA 5.2: *Let* $N_i = \binom{n-d+i-1}{i}$. *For* $j = 0, 1, \ldots, d-1$,

$$h'_{j+1} \le N_{j+1} - \sum_{i=1}^{j} \frac{N_{j+1}}{N_i} \beta_{i-1}.$$

If equality is attained then $\beta_0 = \beta_1 = \cdots = \beta_{j-1} = 0$.

To prove Lemma 5.2 we will use a weaker version of Macaulay's theorem. It asserts that if $a = \binom{x}{r}$ (here $a \in \mathbf{N}$, $x \in \mathbf{R}$) then $a^{<r>} \le \binom{x+1}{r+1}$. In particular,

$$a^{<r>}/a \le \binom{x+1}{r+1} / \binom{x}{r} = (x+1)/(r+1).$$

Since the function $(x+1)/(r+1)$ is increasing as a function of x, we obtain that if $a \le N_r = \binom{n-d+r-1}{r}$ then

$$a^{<r>} \le \frac{n-d+r}{r+1} a = \frac{N_{r+1}}{N_r} a, \tag{17}$$

and equality is attained iff $a = N_r$.

Proof of Lemma 5.2: For $j = 0$ the lemma follows from (5). For $j \ge 1$ we obtain from Theorem 1.7 that

$$\begin{aligned} h'_{j+1} &\le \left(h'_j - \binom{d-1}{j} \beta_{j-1} \right)^{<j>} \\ &\le \left(\left(h'_{j-1} - \binom{d-1}{j-1} \beta_{j-2} \right)^{<j-1>} - \binom{d-1}{j} \beta_{j-1} \right)^{<j>} \\ &\le \cdots \le \left(\cdots \left(\left(h'_1 - \binom{d-1}{1} \beta_0 \right)^{<1>} - \binom{d-1}{2} \beta_1 \right)^{<2>} - \cdots \right)^{<j>} \end{aligned}$$

(applying (17) j times, we obtain)

$$\begin{aligned} &\le N_{j+1} - \sum_{i=1}^{j} \frac{N_{j+1}}{N_j} \frac{N_j}{N_{j-1}} \cdots \frac{N_{i+1}}{N_i} \binom{d-1}{i} \beta_{i-1} \\ &= N_{j+1} - \sum_{i=1}^{j} \frac{N_{j+1}}{N_i} \binom{d-1}{i} \beta_{i-1} \end{aligned}$$

and if equality is attained then $\beta_0 = \beta_1 = \cdots = \beta_{j-1} = 0$. ∎

In the rest of this section we will obtain various bounds on the Betti numbers of homology manifolds. Since the proofs in the odd-dimensional and even-dimensional cases are very similar, we will consider only the even-dimensional case, leaving the details of proofs in the odd-dimensional case to the reader.

LEMMA 5.3: *Let Δ be a $(d-1)$-dimensional homology manifold on n vertices.*

1. *If $d = 2k+1$ then*

$$\beta_k + \frac{n-3k-2}{2k+1}\beta_{k-1} + \sum_{i=1}^{k-1} \frac{\binom{n-2}{2k-i}}{\binom{n-2}{k}} \cdot \frac{n-2k-2}{2k+1}\beta_{i-1} \leq \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}.$$

If equality is attained then $\beta_i = 0$ for $i = 0, 1, \ldots, k-1$ and $\beta_k = \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}$.

2. *If $d = 2k$ then*

$$\beta_{k-1} + \frac{n^2-(2nk+k^2+n+k)}{k(k+2+n)}\beta_{k-2} + \sum_{i=1}^{k-2} \frac{\binom{n-2}{2k-i-1}}{\binom{n-2}{k}} \cdot \frac{n(n-2k-1)}{(n+k+2)k}\beta_{i-1} \leq \frac{\binom{n-k-2}{k}}{\binom{2k-1}{k}} \cdot \frac{n}{n+k+2}.$$

If equality is attained then $\beta_i = 0$ for $i = 0, 1, \ldots, k-2$.

Proof (for $d = 2k+1$): Let $z = N_k - h'_k$. By Lemma 5.2

$$z \geq \sum_{i=1}^{k-1} \frac{N_k}{N_i}\binom{2k}{i}\beta_{i-1} \tag{18}$$

(and equality implies that $\beta_0 = \beta_1 = \cdots = \beta_{k-2} = 0$). Therefore,

$$\begin{aligned} h'_{k+1} &\overset{\text{by Th 1.7}}{\leq} \left(h'_k - \binom{2k}{k}\beta_{k-1}\right)^{<k>} \\ &= \left(N_k - \left(z + \binom{2k}{k}\beta_{k-1}\right)\right)^{<k>} \\ &\overset{\text{by (17)}}{\leq} N_{k+1} - \frac{N_{k+1}}{N_k}z - \frac{N_{k+1}}{N_k}\binom{2k}{k}\beta_{k-1} \end{aligned} \tag{19}$$

(and equality implies that $\beta_0 = \beta_1 = \cdots = \beta_{k-1} = 0$). And, thus,

$$\begin{aligned} \binom{2k+1}{k}(\beta_k - \beta_{k-1}) &\overset{\text{by Lemma 5.1}}{=} h'_{k+1} - h'_k \\ &\overset{\text{by (19)}}{\leq} (N_{k+1} - N_k) \\ &\quad - \left(\frac{N_{k+1}}{N_k} - 1\right)z - \frac{N_{k+1}}{N_k}\binom{2k}{k}\beta_{k-1}, \end{aligned}$$

or, equivalently,

$$\beta_k + \left(\frac{N_{k+1}}{N_k}\frac{\binom{2k}{k}}{\binom{2k+1}{k}} - 1\right)\beta_{k-1} + \frac{N_{k+1}-N_k}{N_k\binom{2k+1}{k}}z \leq \frac{N_{k+1}-N_k}{\binom{2k+1}{k}} = \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}. \tag{20}$$

Substituting the lower bound (18) for z in (20) and simplifying the coefficients of Betti numbers, we obtain the required inequality. ∎

LEMMA 5.4: *If $n \leq \lfloor 3d/2 \rfloor + 2$, then $\beta_{\lfloor (d-1)/2 \rfloor} \geq \beta_{\lfloor (d-1)/2 \rfloor - 1} \geq \cdots \geq \beta_0$.*

Proof: By Lemma 5.1, for any $j = 1, 2, \ldots, \lfloor (d-1)/2 \rfloor$

$$\beta_j - \beta_{j-1} = \frac{h'_{d-j} - h'_j}{\binom{d}{j}} \geq \frac{1 - h'_j}{\binom{d}{j}} \geq \frac{1 - \binom{n-d+j-1}{j}}{\binom{d}{j}} > -1,$$

since $n - d + j - 1 \leq d$ for $n \leq \lfloor 3d/2 \rfloor + 2$, $j \leq \lfloor (d-1)/2 \rfloor$. Thus, $\beta_j - \beta_{j-1} > -1$. Since Betti numbers are integers, we obtain that $\beta_j \geq \beta_{j-1}$ for $j = 1, 2, \ldots, \lfloor (d-1)/2 \rfloor$. ∎

LEMMA 5.5:

(1) *If $n \leq \lfloor 3d/2 \rfloor + 2$, then $\beta_0 = \beta_1 = \cdots = \beta_{\lfloor (d-1)/2 \rfloor - 1} = 0$, and if $n < \lfloor 3d/2 \rfloor + 2$, then also $\beta_{\lfloor (d-1)/2 \rfloor}$ is equal to zero.*

(2) *If $\lfloor 3d/2 \rfloor + 3 \leq n \leq 2d - 2$, then $\beta_0 = \beta_1 = \cdots = \beta_{2d-1-n} = 0$.*

Remark: In the case where Δ is a combinatorial manifold, a similar and somewhat stronger result was proved by Brehm and Kühnel [3].

Proof (for $d = 2k + 1$): First, note that $n \geq d + 1 = 2k + 2$, since Δ is a $(d-1)$-dimensional homology manifold.

- $2k + 2 = d + 1 \leq n \leq \lfloor 3d/2 \rfloor + 1 = 3k + 2$. Then

$$\frac{n - 3k - 2}{2k + 1} \geq -\frac{k}{2k + 1} > -\frac{1}{2}.$$

  By Lemma 5.4, $\beta_k \geq \beta_{k-1}$, so we have from Lemma 5.3 that

$$\frac{1}{2}\beta_k \leq \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}} \overset{n \leq 3k+2}{\leq} \frac{\binom{2k}{k+1}}{\binom{2k+1}{k}} < \frac{1}{2}.$$

  Therefore $\beta_k < 1$, and so $\beta_k = 0$. By Lemma 5.4 we are done.

- $n = \lfloor 3d/2 \rfloor + 2 = 3k + 3$. It follows from Lemma 5.3 that

$$\beta_k + \frac{1}{2k + 1}\beta_{k-1} \leq \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}} = \frac{\binom{2k+1}{k+1}}{\binom{2k+1}{k}} = 1.$$

  Thus, either $\beta_k = 0$ and then by Lemma 5.4 we are done, or $\beta_k = 1$, $\beta_{k-1} = 0$ and then by Lemma 5.4, $\beta_{k-1} = \beta_{k-2} = \cdots = \beta_0 = 0$.

- $3k+4 = \lfloor 3d/2 \rfloor + 3 \leq n \leq 2d-2 = 4k$. From Lemma 5.3 it follows that, for $i \leq k-1$,

$$\beta_{i-1} < \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}} \cdot \frac{\binom{n-2}{k}}{\binom{n-2}{2k-i}} \cdot \frac{2k+1}{n-2k-2} = \frac{\binom{n-2k+i-2}{i}}{\binom{2k}{i}}.$$

Note that if $i \leq 2d-n = 4k-n+2$, then $n-2k+i-2 \leq 2k$, and therefore $\beta_{i-1} < 1$. So for such i, $\beta_{i-1} = 0$. □

Now we are ready to prove the Kühnel conjecture for $2k$-dimensional homology manifolds with at least $4k+3$ or at most $3k+3$ vertices:

THEOREM 5.6: *Let Δ be a $2k$-dimensional homology manifold with n vertices, where $n \geq 4k+3$ or $n \leq 3k+3$. Then*

$$(-1)^k(\chi(\Delta)-2) \leq \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}.$$

(Note, that $(-1)^k(\chi - 2) = \beta_k - (\beta_{k+1} + \beta_{k-1}) + (\beta_{k+2} + \beta_{k-2}) - \cdots = \beta_k + 2\sum_{i=1}^{k}(-1)^i \beta_{k-i}$ by Poincaré duality.)

We prove, in fact, a stronger result:

THEOREM 5.7: *Let Δ be a $(d-1)$-dimensional homology manifold with n vertices.*

(1) *If $d = 2k+1$ and $n \geq 4k+3$ or $n \leq 3k+3$, then*

$$\beta_k + 2\sum_{i=0}^{k-2} \beta_i \leq \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}.$$

(2) *If $d = 2k+1$ and $n \leq 3k+3$ or $n \geq 7k+4$, then*

$$\sum_{i=1}^{2k-1} \beta_i = \beta_k + 2\sum_{i=1}^{k-1} \beta_i \leq \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}.$$

(3) *If $d = 2k$ and $n \leq 3k+2$ or $n \geq 4k+1$, then*

$$\sum_{i=1}^{2k-2} \beta_i = 2\sum_{i=1}^{k-1} \beta_i \leq \frac{\binom{n-k-2}{k}}{\binom{2k-1}{k}} \cdot \frac{2n}{n+k+2}.$$

Proof: Again, we will prove the theorem only in the case of $d = 2k+1$, leaving the case $d = 2k$ to the reader. In this case

- If $n \leq 3(k+1)$, the theorem follows from Lemma 5.3 and Lemma 5.5.

- If $n \geq 4k+3$ then $(n-2k-2)/(2k+1) \geq 1$. Also, for $0 < i \leq k-1$, $2k-i \geq k+1$ and $(2k-i)+(k+1) < 3k+1 < n-2$, so

$$\binom{n-2}{2k-i} \geq \binom{n-2}{k+1}.$$

Therefore,

$$\frac{\binom{n-2}{2k-i}}{\binom{n-2}{k}} \cdot \frac{n-2k-2}{2k+1} \geq \frac{\binom{n-2}{k+1}}{\binom{n-2}{k}} = \frac{n-k-2}{k+1} \geq \frac{3k+1}{k+1} \geq 2 \tag{21}$$

and

$$n-3k-2 > 0. \tag{22}$$

By Lemma 5.3, (21) and (22) we obtain

$$\beta_k + 2\sum_{i=0}^{k-2} \beta_i \leq \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}.$$

- If $n \geq 7k+4$ then

$$\frac{n-3k-2}{2k+1} \geq \frac{4k+2}{2k+1} = 2. \tag{23}$$

And the result follows from Lemma 5.3, (21) and (23). ∎

Remark: Let Δ be a $2k$-dimensional homology manifold with n vertices. The second assertion of Kühnel's conjecture is that if

$$(-1)^k(\chi(\Delta)-2) = \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}$$

then Δ is $(k+1)$-neighborly. Here is the proof of this fact for $n \leq 3k+3$ or $n \geq 4k+3$. From Lemma 5.3, (21) and (22), it follows that for such n

$$(-1)^k(\chi(\Delta)-2) = \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}$$

implies that $\beta_0 = \beta_1 = \cdots = \beta_{k-1} = 0$ and

$$\beta_k = \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}.$$

Then by the Dehn–Sommerville relations for homology manifolds (see Lemma 5.1)

$$h'_{k+1} - h'_k = \binom{2k+1}{k}\beta_k = \binom{n-k-2}{k+1} = N_{k+1} - N_k. \tag{24}$$

On the other hand, since $h'_{k+1} \leq (h'_k)^{<k>}$ and since $h'_{k+1} \leq N_{k+1}$, we obtain from (17)

$$(25) \quad h'_{k+1} - h'_k \leq h'_{k+1} - \frac{N_k}{N_{k+1}} h'_{k+1} = \frac{h'_{k+1}}{N_{k+1}}(N_{k+1} - N_k) \leq N_{k+1} - N_k$$

and equality is attained only if $h'_{k+1} = N_{k+1}$.

Comparing (24) and (25), we obtain that $h'_{k+1} = N_{k+1}$. Therefore, $h_i = h'_i = N_i$ for $i = 0, 1, \ldots, k+1$, and so Δ is $(k+1)$-neighborly. ∎

Remark: To prove the Kühnel conjecture for $2k$-dimensional homology manifolds with n vertices, where $n \leq 3k+3$ or $n \geq 4k+3$, we used the following facts: Theorem 1.7, which asserts that if Δ is a $(d-1)$-dimensional Buchsbaum complex on n vertices then $h'_0(\Delta) = 1$, $h'_1(\Delta) = n - d$ and

$$(26) \quad h'_{r+1}(\Delta) \leq \left(h'_r(\Delta) - \binom{d-1}{r}\beta_{r-1}(\Delta)\right)^{<r>} \qquad \text{for } r = 1, 2, \ldots, d-1;$$

Poincaré's duality theorem and the following version of the Dehn Sommerville relations:

$$(27) \qquad h'_{d-j} - h'_j = \binom{d}{j}(\beta_j - \beta_{j-1}) \quad \text{for } i = 0, 1, \ldots, \lfloor \frac{d}{2} \rfloor.$$

Unfortunately, these facts do not imply the Kühnel conjecture for $2k$-dimensional homology manifolds with n vertices, where $3k+3 < n < 4k+3$. Indeed, there exist numbers k and n such that $3k+3 < n < 4k+3$, and sequences of non-negative numbers $(h'_0, h'_1, \ldots, h'_{2k+1})$ and $(\beta_1, \ldots, \beta_{2k-1})$, where $\beta_i = \beta_{2k-i}$ for $i = 1, 2, \ldots, 2k-1$ and $\beta_0 = 0$, which satisfy (26) and (27), but such that

$$(-1)^k \sum_{i=0}^{2k-1} (-1)^i \beta_i > \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}.$$

For example, let us take $k = 18$ (so $d = 2k+1 = 37$), $n = 62$,

$$\beta_i = \begin{cases} 3 & i = 16, 20 \\ 20 & i = 18 \\ 0 & \text{otherwise} \end{cases}$$

and define h'_r as follows:

$$h'_r = \begin{cases} \binom{n-d+r-1}{r} = \binom{24+r}{r}, & r \leq k-1 = 17, \\ \left(h'_{k-1} - \binom{2k}{k-1}\beta_{k-2}\right)^{<k-1>} = 288985623780, & r = k = 18, \\ h'_k + \binom{2k}{k}\beta_k = 642438261780, & r = k+1 = 19, \\ h'_{k-1} - \binom{2k}{k-1}\beta_{k-2} = 103868374320, & r = k+2 = 20, \\ h'_{k-2} + \binom{2k}{k-2}\beta_{k-2} = 101479425660, & r = k+3 = 21, \\ h'_{d-r} = h'_{37-r}, & r \geq k+4 = 22. \end{cases}$$

In particular $\{h'_i\}$, $\{\beta_i\}$ satisfy (27). A computer check shows that these sequences also satisfy (26), but

$$(-1)^k \sum_{i=0}^{2k-1} (-1)^i \beta_i = 26,$$

whereas

$$\left\lfloor \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}} \right\rfloor = 25.$$

6. Proof of the UBC and the UBC′ for even-dimensional homology manifolds

In this section we prove the UBC for $2k$-dimensional Eulerian homology manifolds and the UBC′ for arbitrary homology manifolds. In fact, we prove that the UBC holds for a more general class of homology manifolds.

Let $d = 2k+1$ and let Δ be a $2k$-dimensional homology manifold on n vertices.

LEMMA 6.1: *For $l = 0, 1, \ldots, 2k+1$*

$$f_{l-1}(\Delta) = \sum_{j=0}^{k+1} a_j^l h_j(\Delta),$$

where a_j^l are non-negative (rational) coefficients independent of Δ.

Proof:

$$\begin{aligned} f_{l-1}(\Delta) &\overset{\text{by def 1.6}}{=} \sum_{j=0}^{d} \binom{d-j}{d-l} h_j \\ &= \sum_{j=0}^{k+1} \binom{d-j}{d-l} h_j + \sum_{j=k+2}^{d} \binom{d-j}{d-l} h_j \\ &\overset{\text{by (1)}}{=} \sum_{j=0}^{k+1} \binom{d-j}{d-l} h_j \\ &\quad + \sum_{j=k+2}^{d} \binom{d-j}{d-l} \left(h_{d-j} + \binom{d}{j} (-1)^{d-j} (\chi(\Delta) - 2) \right) \end{aligned}$$

$$=\sum_{j=0}^{k+1}\binom{d-j}{d-l}h_j+\sum_{j=0}^{k-1}\binom{j}{d-l}\left(h_j+\binom{d}{j}(-1)^j(\chi-2)\right)$$

$$=\sum_{j=0}^{k-1}\left(\binom{d-j}{d-l}+\binom{j}{d-l}\right)h_j+\binom{k+1}{d-l}h_k+\binom{k}{d-l}h_{k+1}$$

$$+(\chi-2)\sum_{j=0}^{k-1}(-1)^j\binom{d}{j}\binom{j}{d-l}$$

$$\overset{\text{by (1)}}{=}\sum_{j=0}^{k-1}\left(\binom{d-j}{d-l}+\binom{j}{d-l}\right)h_j+\binom{k+1}{d-l}h_k+\binom{k}{d-l}h_{k+1}$$

$$+\frac{(-1)^k(h_{k+1}-h_k)}{\binom{d}{k}}\binom{d}{l}\sum_{j=0}^{k-1}(-1)^j\binom{l}{d-j}.$$

Thus, for $j\le k-1$,

$$a_j^l=\binom{d-j}{d-l}+\binom{j}{d-l}\ge 0\,, \tag{28}$$

$$\begin{aligned}a_k^l&=\binom{k+1}{d-l}+\frac{\binom{d}{l}}{\binom{d}{k}}\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{l}{d-j}\\&\overset{\text{by Lemma 3.1}}{=}\binom{k+1}{d-l}+\frac{\binom{d}{l}}{\binom{d}{k}}\binom{l-1}{k+1}\ge 0,\end{aligned} \tag{29}$$

$$\begin{aligned}a_{k+1}^l&=\binom{k}{d-l}-\frac{\binom{d}{l}}{\binom{d}{k}}\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{l}{d-j}\\&\overset{\text{by Lemma 3.1}}{=}\binom{k}{d-l}-\frac{\binom{d}{l}}{\binom{d}{k}}\binom{l-1}{k+1}\\&=\binom{k}{2k+1-l}-\frac{k}{l}\binom{k-1}{2k+1-l}\ge 0.\end{aligned} \tag{30}$$

∎

Remarks: (1) Note that the coefficients a_j^l are rationals, not integers in general.

(2) Notice that the proof of the Lemma follows from the Dehn–Sommerville relations *only*! Therefore, this result holds for a larger class of simplicial complexes, namely, all simplicial complexes Δ satisfying $\chi(\operatorname{lk}F)=1+(-1)^{\dim \operatorname{lk}F}$ for any non-empty face F of Δ. Since the span of f-vectors of all such $2k$-dimensional simplicial complexes has dimension $k+1$ (see [7]) this kind of representation of f-vector in terms of h-vector is unique.

Definition 6.2: For $d = 2k+1$ define

$$M_l(n,d) = \sum_{j=0}^{k+1} a_j^l N_j = \sum_{j=0}^{k+1} a_j^l \binom{n-d+j-1}{j}.$$

Remark 6.3: If Δ is a Eulerian homology $2k$-manifold, then (by the Dehn–Sommerville relations (1)) $h_i = h_{2k+1-i}$ for $i = 0, 1, \ldots, 2k+1$, and we obtain from (28)–(30) (or, directly from the definition of the h-vector) that

$$f_{l-1}(\Delta) = \sum_{j=0}^{k} \left(\binom{2k+1-j}{2k+1-l} + \binom{j}{2k+1-l} \right) h_j(\Delta).$$

We recall that $h_j(C_{2k+1}(n)) = N_j$ for $j \le k$, so

$$f_{l-1}(C_{2k+1}(n)) = \sum_{j=0}^{k} \left(\binom{2k+1-j}{2k+1-l} + \binom{j}{2k+1-l} \right) N_j.$$

Let Δ be a $2k$-dimensional homology manifold on n vertices.

LEMMA 6.4: *$h_j \le N_j$ for $j = 0, 1, \ldots, 4k+2-n$.*

Proof: If $n > 4k$ there is nothing to prove, and if $n \le 4k$ then the proof follows from Lemma 5.5 and (7).

LEMMA 6.5:

$$h_{j+1} \le N_{j+1} \quad \text{for } \max\{0, 4k+2-n\} \le j \le 2k.$$

Proof: Let $i_0 = \max\{0, 4k+2-n\}$. By Lemma 5.2 we have

$$h'_{j+1} \le N_{j+1} - \sum_{i=1}^{j} \frac{N_{j+1}}{N_i} \binom{2k}{i} \beta_{i-1}. \tag{31}$$

Now, observe that for $j \ge i_0$ and $0 \le i \le j$

$$\frac{\frac{N_{j+1}}{N_i}\binom{2k}{i}}{\binom{2k+1}{j+1}} = \frac{\binom{n-2k-1+j}{j+1}\binom{2k}{i}}{\binom{n-2k-2+i}{i}\binom{2k+1}{j+1}} = \frac{n-(2k+1)+j}{2k+1} \cdot \frac{\binom{n-2}{2k-i}}{\binom{n-2}{2k-j}} \ge 1.$$

[The last inequality follows from

- $n-(2k+1)+j \ge 2k+1$, since $(4k+2)-j \le 4k+2-i_0 \le n$.
- $\binom{n-2}{2k-i} \ge \binom{n-2}{2k-j}$, since $0 \le i \le j$, and so $2k-i \ge 2k-j$ and $(2k-i)+(2k-j) \le 4k-j \le 4k-i_0 \le n-2$.]

Thus,

$$\frac{N_{j+1}}{N_i}\binom{2k}{i} \geq \binom{2k+1}{j+1} \quad \text{for } j \geq \max\{0, 4k+2-n\} \text{ and } i \leq j. \tag{32}$$

Combining (31) and (32), we obtain that

$$h'_{j+1} \leq N_{j+1} - \binom{2k+1}{j+1}\sum_{i=1}^{j}\beta_{i-1} \quad \text{for } j \geq \max\{0, 4k+2-n\}. \tag{33}$$

Therefore, for $j \geq \max\{0, 4k+2-n\}$

$$\begin{aligned} h_{j+1} &\overset{\text{by (4)}}{=} h'_{j+1} - \binom{2k+1}{j+1}\sum_{i=0}^{j}(-1)^{j-i-1}\beta_{i-1} \\ &\overset{\text{by (33)}}{\leq} N_{j+1} - 2\binom{2k+1}{j+1}(\beta_{j-1} + \beta_{j-3} + \beta_{j-5} + \cdots) \leq N_{j+1}. \end{aligned}$$ ∎

Now we are ready to prove the UBC and UBC′ .

THEOREM 6.6: *If Δ is a $2k$-dimensional homology manifold, then*

$$h_j(\Delta) \leq N_j \quad \text{for } j = 0, 1, \ldots, 2k+1.$$

In particular, if Δ is Eulerian, then

$$f_{l-1}(\Delta) \leq f_{l-1}(C_d(n)) \quad \text{for } l = 0, 1, \ldots, 2k+1$$

(if equalities are attained in all these inequalities then Δ is k-neighborly and $\beta_i = 0$ for all $i \leq k-2$); and if Δ is non-Eulerian, then

$$f_{l-1}(\Delta) \leq M_l(n, d) \quad \text{for } l = 0, 1, \ldots, 2k+1$$

(if equalities are attained in all these inequalities then Δ is $(k+1)$-neighborly and $\beta_i = 0$ for all $i \leq k-1$ and $\beta_k = \binom{n-k-2}{k+1}/\binom{2k+1}{k}$).

Proof: The first assertion follows from Lemmas 6.4 and 6.5. The second assertion follows from the first one by the definition of $M_l(n,d)$, Lemma 6.1 and Remark 6.3. Now, suppose that equalities are attained in all the inequalities. Then it follows from the first assertion that $h_i = N_i$ for $i = 0, 1, \ldots, k$ in the Eulerian case and $h_i = N_i$ for $i = 0, 1, \ldots, k+1$ in the non-Eulerian case. And so Δ is k-neighborly in the first case and $(k+1)$-neighborly in the second case. Now, let i_0 be the smallest index such that $\beta_{i_0} \neq 0$. Then by (7), $h_{i_0+2} < N_i$, and so

$i_0 \geq k-1$ in the first case and $i_0 \geq k$ in the second case. In addition, we obtain in the second case, that

$$\beta_k = (-1)^k(\chi-2) = (h_{k+1}-h_k)/\binom{2k+1}{k} = (N_{k+1}-N_k)/\binom{2k+1}{k} = \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}},$$

which completes the proof. ∎

Remark: Lemmas 6.4 and 6.5 with suitable notational changes and the same proofs hold for odd-dimensional manifolds as well, thus giving the second proof of the UBC for odd-dimensional homology manifolds.

In fact, the ordinary UBC holds for a more general class of homology manifolds:

THEOREM 6.7: *Let Δ be a $2k$-dimensional homology manifold, such that $\beta_k \leq \sum\{\beta_i\colon i \neq k-2, k, k+2 \text{ and } 1 \leq i \leq 2k-1\}$, or, equivalently,*

$$\beta_k \leq 2\beta_{k-1} + 2\sum_{i=0}^{k-3}\beta_i. \tag{34}$$

Then

$$f_{l-1}(\Delta) \leq f_{l-1}(C_{2k+1}(n)) \quad \text{for } l = 0, 1, \ldots, 2k+1.$$

If equalities are attained in all these inequalities, then $\beta_i = 0$ for $i \leq k-2$.

Proof: If $n \leq 3k+3$, then, by Lemma 5.5, $\beta_0 = \beta_2 = 1 = \cdots = \beta_{k-1} = 0$, so we have from (34) that $\beta_k = 0$. Therefore, Δ is Eulerian and we are done by Theorem 6.6. Thus, suppose $n \geq 3k+4$.

From Lemma 6.1 and Remark 6.3 it follows that it is sufficient to prove that $h_j(\Delta) \leq N_j$ for $j = 0, 1, \ldots, k$ and $h_{k+1}(\Delta) \leq N_k$. By Theorem 6.6 for arbitrary $2k$-dimensional homology manifold Δ, $h_j(\Delta) \leq N_j$ for $j = 0, 1, \ldots, k$. So it suffices to prove that if Δ satisfies (34), then $h_{k+1}(\Delta) \leq N_k$.

1. $n = 3k+4$ or $n = 3k+5$.
   Then $4k - n + 1 \geq k - 4$, so by Lemma 5.5

$$\beta_{k-4} = \beta_{k-5} = \cdots = \beta_1 = 0. \tag{35}$$

   Since, in addition, $k - 1 \geq \max\{4k+2-n, 0\}$, we obtain from (33) that

$$h'_k \leq N_k - \binom{2k+1}{k}(\beta_{k-2} + \beta_{k-3}). \tag{36}$$

Therefore,

$$\begin{aligned} h_{k+1} &\overset{\text{by (4)and(35)}}{=} h'_{k+1} - \binom{2k+1}{k+1}(\beta_{k-1} - \beta_{k-2} + \beta_{k-3}) \\ &\overset{\text{by Lemma 5.1}}{=} h'_k - \binom{2k+1}{k}(-\beta_k + 2\beta_{k-1} - \beta_{k-2} + \beta_{k-3}) \\ &\overset{\text{by (36)}}{\leq} N_k - \binom{2k+1}{k}(-\beta_k + 2\beta_{k-1} + 2\beta_{k-3}) \\ &\overset{\text{by (34) and (35)}}{\leq} N_k. \end{aligned}$$

Thus in this case the theorem is proved.

2. $n \geq 3k+6$.
   Then for $0 \leq i \leq k-3$ (so we assume that $k \geq 3$)

$$\begin{aligned} \frac{\frac{N_k}{N_i}\binom{2k}{i}}{\binom{2k+1}{k}} &= \frac{n-k-2}{2k+1} \cdot \frac{\binom{n-2}{2k-i}}{\binom{n-2}{k+1}} \geq \frac{2k+4}{2k+1} \cdot \frac{\binom{n-2}{k+3}}{\binom{n-2}{k+1}} \\ &= \frac{(2k+4)(n-k-3)(n-k-4)}{(2k+1)(k+3)(k+2)} \geq \frac{(2k+4)(2k+3)(2k+2)}{(k+2)(k+3)(2k+1)} \\ &> 2 \cdot \frac{2k+3}{k+3} \geq 3. \end{aligned}$$

Thus

$$\frac{N_k}{N_i}\binom{2k}{i} \geq 3\binom{2k+1}{k} \quad \text{for } i \leq k-3. \tag{37}$$

Since $k-1 \geq \max\{0, 4k+2-n\}$, we observe from (32) that

$$\frac{N_k}{N_i}\binom{2k}{i} \geq \binom{2k+1}{k} \quad \text{for } i = k-2, k-1. \tag{38}$$

From Lemma 5.2, (38) and (37) we obtain

$$h'_k \leq N_k - \binom{2k+1}{k}(\beta_{k-2} + \beta_{k-3} + 3\sum_{i=0}^{k-4}\beta_i). \tag{39}$$

Therefore,

$$\begin{aligned}
h_{k+1} &\overset{\text{by (4)}}{=} h'_{k+1} - \binom{2k+1}{k+1}(\beta_{k-1} - \beta_{k-2} + \beta_{k-3} - \cdots) \\
&\overset{\text{by Lemma 5.1}}{=} h'_k - \binom{2k+1}{k}(-\beta_k + 2\beta_{k-1} \\
&\qquad + (-\beta_{k-2} + \beta_{k-3} - \beta_{k-4} + \beta_{k-5} - \cdots)) \\
&\overset{\text{by (39)}}{\leq} N_k - \binom{2k+1}{k}(-\beta_k + 2\beta_{k-1} + 2\beta_{k-3} \\
&\qquad + (2\beta_{k-4} + 4\beta_{k-5} + 2\beta_{k-6} + 4\beta_{k-7} + \cdots)) \\
&\leq N_k - \binom{2k+1}{k}(-\beta_k + 2\beta_{k-1} + 2\sum_{i=0}^{k-3}\beta_i) \overset{\text{by (34)}}{\leq} N_k.
\end{aligned}$$

Discussion of equality is exactly the same as in the proof of Theorem 6.6. ∎

COROLLARY 6.8: *The ordinary UBC holds for the following classes of homology manifolds:*

1. *homology $2k$-manifolds with vanishing middle homology;*
2. *homology $2k$-manifolds with $\chi \leq 2$ when k is even or with $\chi \geq 2$ when k is odd.*

Proof: Such homology manifolds satisfy the assumptions of Theorem 6.7. ∎

7. Further remarks and conjectures

Here we consider some conjectures concerning the combinatorial structure of the shifted basis of the Stanley–Reisner ring of a (triangulation of) topological manifold and relations between the face numbers and the Betti numbers of triangulated manifolds to which these conjectures lead.

First, a brief summary of what we have seen in Section 4. Let Δ be a $(d-1)$-dimensional simplicial Buchsbaum complex on n vertices. Let R_Δ be its Stanley–Reisner ring. Consider generic linear forms $y_1, y_2, \ldots, y_n$, order monomials in the y's lexicographically and choose from these monomials a basis of R_Δ over $\mathbf{k}$, $S(\Delta)$, in the greedy way. For $i = 0, 1, \ldots, d-1$ define A_i to be the set of all monomials m in $y_{i+1}, y_{i+2}, \ldots, y_n$, such that m is in $S(\Delta)$, $y_i m$ is in $S(\Delta)$, but $y_{i+1}m$ is *not* in $S(\Delta)$, and define A_d to be the set of all monomials m in variables $y_{d+1}, \ldots, y_n$ such that $y_d m$ is in $S(\Delta)$. Let MON(i) be the set of all monomials in $y_i, \ldots, y_n$. Let $\tilde{S}(\Delta) = S(\Delta) \cap \text{MON}(d+1)$:

We proved that for $i = 0, 1, \ldots, d-1$

1. There are exactly $\binom{i-1}{r-1}\beta_{r-1}$ monomials m of degree r in $y_{i+1}, y_{i+2}, \dots, y_n$ such that $m \in S(\Delta)$, $y_i m \in S(\Delta)$, but $y_{i+1}m \notin S(\Delta)$.

2. All these monomials are actually monomials in $y_{d+1}, \dots, y_n$.

3. If $y_{i+1}^2 m \notin S(\Delta)$ then $y_{i+1}m \notin S(\Delta)$.

These facts allowed us to calculate the generating function of A_d. We obtained (see Lemma 4.2) that the number of monomials in A_d of degree r is $h'_r - \binom{d-1}{r}\beta_{r-1}$. Since $\partial_{r+1}\tilde{S}_{r+1}(\Delta) \subset A_d$, this led to the proof of Theorem 1.7, which asserts that

$$h'_{r+1}(\Delta) \leq \left(h'_r(\Delta) - \binom{d-1}{r}\beta_{r-1}(\Delta)\right)^{<r>} \qquad \text{for } r = 1, 2, \dots, d-1.$$

Gil Kalai conjectured that if Δ is a triangulation of a topological manifold (briefly, triangulated manifold), then stronger results hold. To state his conjectures let us define $C_{d+1} = \{m \in A_d \cap \mathrm{MON}(d+2) : y_{d+1}m \notin S(\Delta)\}$. That is, C_{d+1} is the set of all monomials m in variables $y_{d+2}, \dots, y_n$, such that $y_d m \in S(\Delta)$, but $y_{d+1}m \notin S(\Delta)$. Let H_r be the set of all monomials of degree r in $A_d - C_{d+1}$. Then $\partial_{r+1}\tilde{S}_{r+1}(\Delta) \subset H_r$, rather than in A_d, as proven.

CONJECTURE 7.1: *Let Δ be a $(d-1)$-dimensional triangulated manifold (with or without boundary, orientable or not orientable). Then the number of monomials in C_{d+1} of degree r is equal to $\binom{d-1}{r-1}\beta_{r-1}$.*

Remark 7.2: Lemma 4.2 and Conjecture 7.1 imply that

$$|H_r| = h'_r - \binom{d-1}{r}\beta_{r-1} - \binom{d-1}{r-1}\beta_{r-1} = h'_r - \binom{d}{r}\beta_{r-1}$$

for $1 \leq r < d$.

Define

$$h''_r = h'_r - \binom{d}{r}\beta_{r-1} = h_r + \binom{d}{r}\sum_{i=1}^{r}(-1)^i\beta_{r-i}$$

(the last equality is the Schenzel theorem).

From now on, we will consider only triangulated manifolds without boundary (and orientable over $\mathbf{k}$).

The following lemma is another version of the Dehn–Sommerville relations:

LEMMA 7.3: *For Δ as above, $h''_i = h''_{d-i}$ for $i = 0, 1, \dots, d$.*

Proof:

$$h''_{d-i} - h''_i = \left(h'_{d-i} - \binom{d}{i}\beta_{d-i-1}\right) - \left(h'_i - \binom{d}{i}\beta_{i-1}\right)$$

$$\overset{\text{by Lemma 5.1}}{=} \binom{d}{i}(\beta_i - \beta_{i-1}) - \binom{d}{i}(\beta_{d-i-1} - \beta_{i-1})$$

$$\overset{\text{by Poincaré duality}}{=} 0. \quad \blacksquare$$

Remark 7.4: h'' can be regarded as the "correct" h-vector for triangulated manifolds without boundary.

CONJECTURE 7.5: *If Δ is a $(d-1)$-dimensional triangulated manifold then*

1. *If $r < d/2$ and $m \in H_r$, then $y_{d+1}^{d-2r} m \in H_{d-r}$ (Conjecture 7.1 and Lemma 7.3 imply that $m \mapsto y_{d+1}^{d-2r} m$ is a bijection).*
2. *If $r < d/2$, then $|\{m \in H_r \cap \mathrm{MON}(d+2)\}| \geq \binom{d}{r-1}\beta_{r-1}$.*

Remark: The first part of Conjecture 7.5 is the analog of the Hard–Lefschetz theorem. If it is true, it implies that $h''_0 \leq h''_1 \leq \cdots \leq h''_{\lfloor d/2 \rfloor}$ and that $h''_{i+1} - h''_i \leq (h''_i - h''_{i-1})^{<i>}$ for $i = 0, 1, \ldots, \lfloor d/2 \rfloor - 1$.

The second part of Conjecture 7.5 provides lower bounds on $h''_{i+1} - h''_i$ for $i = 0, 1, \ldots, \lfloor d/2 \rfloor - 1$.

It is interesting to clarify for what families of simplicial complexes the UBC holds. For example, it is possible that the UBC holds for all simplicial complexes Δ, such that every link Δ' (of face) of dimension $2r$ ($r = 1, 2, \ldots$) satisfies

$$\beta_r \leq \sum\{\beta_i \colon i \neq r-2, r, r+2, 1 \leq i \leq 2r-1\},$$

and, in particular, if the Betti numbers of all links vanish in the middle dimension.

We proved that for even-dimensional homology manifolds, $h_i \leq \binom{n-d+i-1}{i}$ for any $i \leq d$, where $d-1$ is the dimension of the manifold (see Theorem 6.6). (And the proof works for odd-dimensional homology manifolds as well.) It is not clear whether these inequalities hold for all Buchsbaum complexes. Wistuba and Ziegler obtained that these inequalities do not hold for some pure simplicial complexes (see [22]).

It is interesting to note that from our proofs it follows that if n is sufficiently large then every homology $2k$-manifold with n vertices such that $\beta_i \neq 0$ for some $i \neq k-1, k, k+1$, $0 < i < 2k$ satisfies the UBT. (In other words, for each such homology manifold there are only finitely many triangulations that violate the UBT.)

THEOREM 7.6: *Let Δ be a $2k$-dimensional triangulated manifold on n vertices. Conjecture 7.1 implies that*

$$\beta_k(\Delta)+\beta_{k-1}(\Delta)+2\sum_{i=0}^{k-2}\beta_i(\Delta)\leq\frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}.$$

In particular, Conjecture 7.1 implies the Kühnel conjecture for any n.

Proof: Since $\partial_{r+1}\tilde{S}_{r+1}(\Delta)\subset H_r$, $|\tilde{S}_{r+1}(\Delta)|=h'_{r+1}$ and, by Remark 7.2, $|H_r|=h'_r-\binom{d}{r}\beta_{r-1}$, we obtain that

$$h'_{r+1}\leq\left(h'_r-\binom{d}{r}\beta_{r-1}\right)^{<r>}. \tag{40}$$

By Lemma 5.5 we can assume that $n\geq 3k+4$. Repeating the calculations of Lemmas 5.2 and 5.3, but using (40) instead of Theorem 1.7, we obtain

$$\beta_k+\frac{n-2k-2}{k+1}\beta_{k-1}+\sum_{i=1}^{k-1}\frac{\binom{n-2}{2k-i}}{\binom{n-2}{k}}\cdot\frac{n-2k-2}{2k+1-i}\beta_{i-1}\leq\frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}. \tag{41}$$

Now, note that for $n\geq 3k+4$ and $i=k-1$

$$\frac{\binom{n-2}{2k-i}}{\binom{n-2}{k}}\cdot\frac{n-2k-2}{2k+1-i}=\frac{(n-k-2)(n-2k-2)}{(k+1)(k+2)}\geq\frac{(2k+2)(k+2)}{((k+1)(k+2)}=2; \tag{42}$$

for $n\geq 3k+4$ and $i\leq k-2$

$$\begin{aligned}\frac{\binom{n-2}{2k-i}}{\binom{n-2}{k}}\cdot\frac{n-2k-2}{2k+1-i}&\geq\frac{\binom{n-2}{k+2}}{\binom{n-2}{k}}\cdot\frac{n-2k-2}{2k+1}\\&=\frac{(n-k-2)(n-k-3)(n-2k-2)}{(k+1)(k+2)(2k+1)}\\&\geq\frac{(2k+2)(2k+1)(k+2)}{(k+1)(k+2)(2k+1)}=2;\end{aligned} \tag{43}$$

and

$$\frac{n-2k-2}{k+1}>1. \tag{44}$$

Substitution of (42)–(44) in (41) completes the proof of the theorem. ∎

Remark 7.7: Let Δ be a $2k$-dimensional triangulated manifold on n vertices, such that $\beta_k(\Delta)=0$. Then Conjecture 7.1 and the first part of Conjecture 7.5 imply that $S(\Delta)\subset S(C_{2k+1}(n))$.

Indeed, $\beta_k = 0$, therefore $|\tilde{S}_{k+1}| = h'_{k+1} = h''_{k+1} = |H_{k+1}|$. Since $H_{k+1} \subset \tilde{S}_{k+1}$, it follows that $H_{k+1} = \tilde{S}_{k+1}$. Thus, by the first part of Conjecture 7.5, all monomials in $\tilde{S}_{k+1}$ are divisible by y_{d+1}. In particular, $y_{d+2}^{k+1} \notin S(\Delta)$. Therefore, for any $m \in S(\Delta)$, m is not divisible by y_{d+2}^{k+1}. Thus, $S(\Delta) \subset S(C_{2k+1}(n))$ (see [6] for the description of $S(C_{2k+1}(n))$). If so, then Δ satisfies the generalized upper bound theorem (see [6]).

In general, if $y_{d+2}^{k+1} \notin S(\Delta)$ then the UBC and some far-reaching generalizations of the UBC hold for Δ (see [6]). In light of Conjecture 7.5, it seems that not all $(d-1)$-manifolds with $\chi = 1 + (-1)^{d-1}$ satisfy this condition. But it is possible that this condition holds for all simplicial complexes Δ such that for every link Δ' (of a face) of dimension $2r$ ($r = 1, 2, \ldots$), $\dim H_r(\Delta', \mathbf{Z}_2) = 0$.

8. Appendix (by Richard Stanley)

Let M be a graded Buchsbaum module of Krull dimension $d > 0$ over the graded algebra R. Let $m = R_+$ be the irrelevant maximal ideal of R. Let $G_i = G_i(t)$ be the Poincaré series of the local cohomology module $H^i(M)$ (in the variable t). Thus G_i is a Laurent polynomial for $0 \le i \le d-1$. Write

$$G(M) = (G_0, G_1, \ldots, G_{d-1}).$$

Recall that $H^0(M)$ is a submodule of M, viz.,

$$H^0(M) = \{u \in M : m^k u = 0 \text{ for some } k \ge 1\}.$$

Lemma 8.1: *We have*

$$H^i(M/H^0(M)) = \begin{cases} 0, & i = 0, \\ H^i(M), & i > 0. \end{cases}$$

Proof: The short exact sequence

$$0 \longrightarrow H^0(M) \longrightarrow M \longrightarrow M/H^0(M) \longrightarrow 0$$

gives rise to the long exact local cohomology sequence

$$\begin{aligned} \cdots \longrightarrow H^i(H^0(M)) \longrightarrow H^i(M) \longrightarrow H^i(M/H^0(M)) \\ \longrightarrow H^{i+1}(H^0(M)) \longrightarrow \cdots. \end{aligned} \tag{45}$$

Since $H^0(M)$ has Krull dimension 0, we have $H^i(H^0(M)) = 0$ if $i > 0$, while $H^0(H^0(M)) = H^0(M)$. The proof follows from (45). ∎

The next lemma appears at the top of page 76 of [20], but without the grading.

Lemma 8.2: *Let $x \in m$ be a homogeneous non-zero divisor (NZD) of degree a on M. For $i < d-1$ we have*

$$H^i(M/xM) = H^i(M) \oplus H^{i+1}(M(-a)),$$

where $M(-a)$ denotes M with the grading shifted by a.

Proof: The short exact sequence

$$0 \longrightarrow M(-a) \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

yields

$$\begin{aligned} \cdots \longrightarrow H^i(M(-a)) \xrightarrow{x} H^i(M) \longrightarrow H^i(M/xM) \\ \longrightarrow H^{i+1}(M(-a)) \longrightarrow \cdots . \end{aligned} \tag{46}$$

Now since M is Buchsbaum, we have $mH^i(M) = 0$ for $i < d$ (see Corollary 2.4 on p. 75 of [20]). Hence the map $H^i(M(-a)) \xrightarrow{x} H^i(M)$ in (46) is 0 for $i < d$, and we get an exact sequence

$$0 \longrightarrow H^i(M) \longrightarrow H^i(M/xM) \longrightarrow H^{i+1}(M(-a)) \longrightarrow 0$$

for $i < d-1$, and the proof follows. ∎

Corollary 8.3: *With x as in Lemma 8.2, we have*

$$G_i(M/xM) = G_i(M) + t^a G_{i+1}(M),$$

for $0 \le i \le d-2$.

Now consider a peeling of M. At first we have

$$G(M) = (G_0, G_1, G_2, \ldots).$$

Let $M_1 = M/H^0(M)$. (Possibly $H^0(M) = 0$, as for face rings, but this is irrelevant.) By Lemma 8.1,

$$G(M_1) = (0, G_1, G_2, \ldots).$$

Let x_1 be an NZD on M_1 of degree one. Let $M_2 = M_1/x_1M_1$. By Corollary 8.3, we get

$$G(M_2) = (tG_1, G_1 + tG_2, G_2 + tG_3, \ldots).$$

Now let $M_3 = M_2/H^0(M_2)$. By Lemma 8.1, we get

$$G(M_3) = (0, G_1 + tG_2, G_2 + tG_3, \ldots).$$

Let x_2 be an NZD on M_3 of degree one. Let $M_4 = M_3/x_2M_3$. By Corollary 8.3, we get

$$G(M_4) = (t(G_1 + tG_2), G_1 + tG_2 + t(G_2 + tG_3), G_2 + tG_3 + t(G_3 + tG_4), \ldots)$$
$$= (tG_1 + t^2G_2, G_1 + 2tG_2 + t^2G_3, G_2 + 2tG_3 + t^2G_4, \ldots).$$

Continuing in this way, we obtain the entire Poincaré series for the peeling of M. In particular, we obtain the expression for $G_0(M_{2k})$, which is the Poincaré series of $H^0(M_{2k})$.

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References

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Mass., 1969.

[2] A. Björner and G. Kalai, *An extended Euler-Poincaré theorem*, Acta Mathematica **161** (1988), 279–303.

[3] U. Brehm and W. Kühnel, *Combinatorial manifolds with few vertices*, Topology **26** (1987), 465–473.

[4] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, Berlin, 1995.

[5] C. Greene and D. J. Kleitman, *Proof techniques in the theory of finite sets*, in *Studies in Combinatorics* (Gian-Carlo Rota, ed.), Volume 17 of *Studies in Mathematics*, Mathematical Association of America, 1978, pp. 22–79.

[6] G. Kalai, *The diameter of graphs of convex polytopes and f-vector theory*, in *Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift* (P. Gritzmann and B. Sturmfels, eds.), Volume 4 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, 1991, pp. 387–411.

[7] V. Klee, *A combinatorial analogue of Poincaré's duality theorem*, Canadian Journal of Mathematics **16** (1964), 517–531.

[8] V. Klee, *The number of vertices of a convex polytope*, Canadian Journal of Mathematics **16** (1964), 702–720.

[9] W. Kühnel, *Triangulations of manifolds with few vertices*, in *Advances in Differential Geometry and Topology* (F. Tricerri, ed.), World Scientific, Institute for Scientific Interchange, 1990, pp. 59–114.

[10] W. Kühnel, *Tight Polyhedral Submanifolds and Tight Triangulations*, Springer-Verlag, Berlin, 1995.

[11] P. McMullen, *The maximum numbers of faces of a convex polytope*, Mathematika **17** (1970), 179–184.

[12] T. S. Motzkin, *Comonotone curves and polyhedra*, Bulletin of the American Mathematical Society **63** (1957), 35.

[13] J. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Reading, Mass., 1984.

[14] G. Ringel, *Map Color Theorem*, Springer-Verlag, Berlin, 1974.

[15] K. S. Sarkaria, *On neighborly triangulations*, Transactions of the American Mathematical Society **227** (1983), 213–239.

[16] P. Schenzel, *On the number of faces of simplicial complexes and the purity of Frobenius*, Mathematische Zeitschrift **178** (1981), 125–142.

[17] R. P. Stanley, *The upper bound conjecture and Cohen-Macaulay rings*, Studies in Applied Mathematics **54** (1975), 135–142.

[18] R. P. Stanley, *Hilbert functions and graded algebras*, Advances in Mathematics **28** (1978), 57–83.

[19] R. P. Stanley, *Combinatorics and Commutative algebra*, Birkhauser, Boston, 1983.

[20] J. Stückrad and W. Vogel, *Buchsbaum Rings and Applications*, Springer-Verlag, Berlin, 1986.

[21] D. Walkup, *The lower bound conjecture for 3- and 4-manifolds*, Acta Mathematica **125** (1970), 75–107.

[22] G. M. Ziegler, *Lectures on polytopes*, Springer-Verlag, Berlin, 1994.